

# FREQUENCY OF DIMENSION DISTORTION UNDER QUASISYMMETRIC MAPPINGS

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**ABSTRACT.** We study the distortion of Hausdorff dimension of families of Ahlfors regular sets under quasisymmetric map  $f$  between metric spaces. We show that  $f$  cannot increase the dimension of “most”  $d$ -regular sets and we estimate the number of exceptional sets whose images have dimension  $\geq d' > d$ ; the precise statements of both results involve modulus estimates for families of measures. For planar quasiconformal maps, the general estimates imply that if  $E \subset \mathbb{R}$  is  $d$ -regular, then some component of  $f(E \times \mathbb{R})$  has dimension  $\leq 2/(d+1)$ , and we construct examples to show this bound is sharp. In addition, we construct 1-dimensional sets  $E$  so that  $f(E \times \mathbb{R})$  contains no rectifiable sub-arc. These results generalize work of Balogh, Monti and Tyson [5] and answer questions posed in [5], [8].

## 1. INTRODUCTION

It is well known that the quasiconformal image of a single line can be a fractal curve (e.g., the von Koch snowflake), even though the absolute continuity of quasiconformal maps implies that “most” lines must have locally rectifiable images. The purpose of this paper is to investigate how many lines can be made into fractal curves by a single map  $f$ . The precise results are given in terms of  $p$ -modulus of measures and will be stated after we have reviewed the necessary definitions. However, the basic idea is easy to state:

**Theorem 1.1.** *Suppose  $E \subset \mathbb{R}^n$  is a closed, Ahlfors  $d$ -regular set and  $Y \subset \mathbb{R}^m$  is a Borel set. If  $f$  is a quasisymmetric map defined on  $E \times Y$  then*

$$(1.1) \quad \frac{\dim_H(E \times Y)}{\dim_H(E)} \leq \frac{\dim_H f(E \times Y)}{\inf_y \dim_H f(E \times \{y\})}.$$

To see the relation to making many lines fractal simultaneously, take  $E = \mathbb{R}$  and  $n = m = 1$  in Theorem 1.1. Since  $\dim f(\mathbb{R} \times Y) \leq 2$ , inequality (1.1) gives

$$\dim_H(\mathbb{R} \times Y) \leq \frac{2}{\inf_{y \in Y} \dim_H f(\mathbb{R} \times \{y\})}.$$

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or, since  $\dim_H(\mathbb{R} \times Y) = 1 + \dim_H Y$ , see e.g Theorem 8.10 in [19],

$$\inf_{y \in Y} \dim_H f(\mathbb{R} \times \{y\}) \leq \frac{2}{1 + \dim_H(Y)}.$$

For example, a quasiconformal map can make every component of  $[0, 1] \times Y$  have dimension  $> 3/2$  only if  $\dim(Y) < 1/3$ . On the other hand, if we set  $Y = \mathbb{R}$  and  $n = m = 1$ , a similar calculation gives

$$(1.2) \quad \inf_{y \in \mathbb{R}} \dim_H f(E \times \{y\}) \leq \frac{2\dim(E)}{1 + \dim(E)}.$$

We will show both estimates are sharp, in particular answering Problem 6.3 of [5] in the planar case.

**Theorem 1.2.** *Let  $E \subset i\mathbb{R}$  be a closed Ahlfors  $d$ -regular set.*

(a) *For a quasiconformal mapping  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  of the plane*

$$\begin{aligned} \inf_{y \in E} \dim_H f(\mathbb{R} + y) &\leq \frac{2}{d+1}, \\ \inf_{x \in \mathbb{R}} \dim_H f(x + E) &\leq \frac{2d}{d+1}. \end{aligned}$$

(b) *For any  $0 < d \leq 1$  and  $\epsilon > 0$ , there are  $f$  and  $E$  as above, so that*

$$\begin{aligned} \inf_{y \in E} \dim_H f(I + y) &\geq \frac{2}{d+1} - \epsilon, \\ \inf_{x \in I} \dim_H f(x + E) &\geq \frac{2d}{d+1} - \epsilon, \end{aligned}$$

*for every non-trivial interval  $I \subset \mathbb{R}$ .*

If  $\dim(E) = 1$ , this implies that  $\inf_{y \in E} \dim(f(\mathbb{R} \times \{y\})) = 1$ . Must there be a  $y$  such that the image of  $\mathbb{R} \times \{y\}$  is rectifiable? The following says no.

**Theorem 1.3.** *For any increasing function  $h$  on  $[0, \infty)$  such that*

$$\lim_{t \rightarrow 0} \frac{h(t)}{t} = \infty,$$

*there is a compact set  $E \subset [0, 1]$  and a quasiconformal map  $f$  so that*

- (1) *The quasiconformal constant of  $f$  is bounded independent of  $h$  and  $E$ .*
- (2)  *$E$  has infinite Hausdorff measure with respect to  $h$ .*
- (3)  *$f([0, 1] + iy)$  contains no rectifiable subarc for any  $y \in E$ .*

In particular, there is a compact set  $E \subset [0, 1]$  of Hausdorff dimension 1 and a quasiconformal map  $f$  of the plane so that  $f([0, 1] + iy)$  contains no rectifiable subarcs for any  $y \in E$ . Previously, it was unknown if there was even an uncountable set  $E$  with this property. In fact, validity of Theorem 1.3 was explicitly asked as Problem 6.4 in [5] and Problem 5.3 in [8]. Kovalev and Onninen proved in [18] that given any countable collection  $\mathcal{L}$  of

parallel lines in  $\mathbb{R}^2$  there is planar quasiconformal image of  $\mathcal{L}$  that contains no rectifiable subarcs.

Rewriting inequality (1.1) as follows

$$(1.3) \quad \inf_{y \in Y} \frac{\dim_H f(E \times \{y\})}{\dim_H(E \times \{y\})} \leq \frac{\dim_H f(E \times Y)}{\dim_H(E \times Y)},$$

we obtain the principle “*fiberwise expansion implies global expansion*”: if every fiber  $E \times \{y\}$  has its dimension increased by a factor  $\lambda \geq 1$ , then the dimension of the whole product  $E \times Y$  increases by at least a factor of  $\lambda$  as well. For instance, if  $E$  is minimal for conformal dimension, i.e. if  $\dim_H f(E) \geq \dim_H E$  for any quasimetric mapping  $f$ , then we see  $E \times Y$  is also minimal for conformal dimension. When  $E = \mathbb{R}$  this gives the well known result of Tyson [23]. Also see the discussion after Theorem 3.1 for much more general results in vein of inequality (1.1).

*Remark 1.4.* We cannot reverse (1.3) by replacing the infimum by a supremum. Consider a quasiconformal map  $f$  that maps  $i\mathbb{R}$  to a curve of dimension  $D > 1$ , but is smooth elsewhere. If  $Y = \mathbb{R}$  and  $E \subset \mathbb{R}$  contains 0 and has dimension  $0 < d < D - 1$ , then the right side of (1.3) is  $\geq D/(d+1) > 1$ , but the smoothness of  $f$  off  $i\mathbb{R}$  implies  $\dim f(E \times \{y\})/\dim(E \times \{y\}) = 1$  for every  $y$ . In other words “*global expansion does not imply fiberwise expansion*”.

This paper is organized as follows. In Section 2 we review the necessary definitions and results needed in the paper. In Section 3 we state our main results in their general form. In Sections 4 and 5 we prove Theorems 3.3, 3.4 and 1.1. Sections 6 and 7 are devoted to the construction of examples of quasiconformal maps in Theorems 1.2 and 1.3. In Section 8 we state several corollaries of the mains results and list some open problems.

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## 2. PRELIMINARIES

**2.1. Measures, dimension and Ahlfors regularity.** Given a metric space  $X$  we will denote by  $B(x, r)$  the open ball in  $X$  of radius  $r$  around  $x$ . Often we will denote by  $B_r$  a ball of radius  $r$  with an unspecified center. Given a non-negative function  $\varphi : [0, \infty) \rightarrow [0, \infty)$ , usually called a *gauge function*, the Hausdorff  $\varphi$ -measure of a metric space  $(X, d_X)$  is defined as follows. For

every open cover of  $X$  by balls  $B(x_i, r_i), i \in \mathbb{N}$  let

$$H_\varphi^\varepsilon(X) = \inf \left\{ \sum_{i=1}^{\infty} \varphi(r_i) : X \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i < \varepsilon \right\},$$

and

$$H_\varphi(X) = \lim_{\varepsilon \rightarrow 0} H_\varphi^\varepsilon(X).$$

When  $\varphi(x) = x^t, t \geq 0$  the resulting measure is called the  $t$  dimensional Hausdorff measure and is denoted by  $H_t$ . It is easy to see that if  $H_t(E) < \infty$  then  $H_s(E) = 0$  for  $t < s$ . Therefore it is natural to define Hausdorff dimension of  $X$  as

$$\dim_H(X) = \inf \{ t : H_t(X) = 0 \}.$$

One usually gives an upper bound for the Hausdorff dimension of a set by finding explicit covers for it. Lower bounds can be obtained by finding a measure on  $X$ .

**Lemma 2.1** (Mass distribution principle). *If the metric space  $(X, d_X)$  supports a positive Borel measure  $\mu$  satisfying  $\mu(U) \leq C(\text{diam}U)^d$ , for some fixed constant  $C > 0$  and every  $U \subset X$  then  $\dim_H(E) \geq d$ .*

An important converse is the following lemma.

**Lemma 2.2** (Frostman's Lemma). *If  $X \subset \mathbb{R}^N$  is a Borel set of Hausdorff dimension  $d$  and  $0 < s < d$  then there is a finite and positive measure  $\mu$  on  $X$  such that  $\mu(B_r) \lesssim r^s$  for every ball  $B_r \subset X$ .*

$E \subset \mathbb{R}^N$  is *Ahlfors  $d$ -regular*, if there is a constant  $C \geq 1$  such that for every  $x \in E$  and  $0 < r < \text{diam}E$  the following inequalities hold

$$(2.1) \quad \frac{1}{C}r^d \leq \mathcal{H}_d(E \cap B(x, r)) \leq Cr^d.$$

A measure  $\mu$  on a metric space  $X$  is *Ahlfors  $p$ -regular* for some  $p > 0$ , if

$$(2.2) \quad C^{-1}r^p \leq \mu(B_r) \leq Cr^p,$$

for some constant  $1 < C < \infty$  and every ball  $B_r \subset X$  of radius  $r < r_0$ . Moreover, we say  $\mu$  is *upper or lower  $p$ -regular* if only the right or left inequality in (2.2) holds, respectively.

For  $d > 0$  we say that a family of measures  $\mathcal{L}_\mathcal{E} = \{\lambda_E\}_{E \in \mathcal{E}}$  associated to a collection of subsets  $\mathcal{E} = \{E\}$  of  $X$  is *Ahlfors  $d$ -regular* if

$$(2.3) \quad C_1^{-1}r^d \leq \lambda_E(B(x, r)) \leq C_1r^d, \quad \forall x \in E$$

for every  $r < r_0$  and some  $C_1 > 0$ , which is independent of  $E \in \mathcal{E}$ . Similarly,  $\mathcal{L}$  is *upper or lower  $d$ -regular* if only the right or left inequality in (2.3) holds, respectively.

We refer to [19] and [12] for proofs of the lemmas above and for further discussion of Hausdorff measures, dimension and Ahlfors regular spaces and their properties.

**2.2. Quasiconformal and quasimetric mappings.** Given a homeomorphism  $f : X \rightarrow Y$ ,  $x \in X$  and  $r > 0$  we let

$$(2.4) \quad H_f(x, r) = \frac{\sup_{|y-x| \leq r} |f(x) - f(y)|}{\inf_{|y-x| \geq r} |f(x) - f(y)|}.$$

The mapping  $f$  is called (metrically) *quasiconformal* if there is a constant  $H < \infty$  such that

$$(2.5) \quad \limsup_{r \rightarrow 0} H_f(x, r) \leq H$$

for every  $x \in X$ .

Because quasiconformality is an infinitesimal property it is often hard to work with directly. For this reason one often requires a stronger, global condition from a mapping  $f$ , which we discuss next.

Let  $\eta : [0, \infty) \rightarrow [0, \infty)$  be a fixed homeomorphism. A homeomorphism  $f$  between metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  is called  $\eta$ -*quasimetric* if for all distinct triples  $x, y, z \in X$  and  $t > 0$

$$(2.6) \quad \frac{d_X(x, y)}{d_X(y, z)} \leq t \quad \Rightarrow \quad \frac{d_Y(f(x), f(y))}{d_Y(f(y), f(z))} \leq \eta(t).$$

Quasimetric mappings do not distort (macroscopic) shapes too much. In particular an image of a round ball will be “roundish”, a condition which a priori holds for QC maps only on small scales depending on  $x$ . More precisely there is a constant  $H \geq 1$  such that for every ball  $B = B(x, R) \subset X$  there is a ball  $B' = B(f(x), r)$  so that the following holds

$$(2.7) \quad \frac{1}{H} \cdot B' \subset f(B) \subset H \cdot B',$$

where we may take  $H = \eta(1)$ .

Even though quasimetricity is a stronger condition than quasiconformality, it is often the case that the two notions coincide. For instance if a homeomorphism  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ ,  $N \geq 2$  is QC then it is also QS. This was proved by Gehring for  $N = 2$  in [10] and by Vaisala for  $N \geq 3$ , [24].

**2.3. Modulus of curves and measures.** The main tool used in this paper is modulus. In this section we define several notions of moduli and record some of their properties which we need below.

**2.3.1. Modulus of a curve family.** Given a metric measure space  $(X, \mu)$ , a family of curves  $\Gamma$  in  $X$  and a real number  $p \geq 1$  the  $p$ -*modulus* of  $\Gamma$  is defined as

$$\text{mod}_p \Gamma = \inf_{\rho} \int_X \rho^p d\mu,$$

where the infimum is taken over all  $\Gamma$ -*admissible* nonnegative Borel functions  $\rho$ . Here a function  $\rho : X \rightarrow [0, \infty)$  is  $\Gamma$  admissible if  $\int_{\gamma} \rho ds \geq 1$  for every locally rectifiable curve  $\gamma \in \Gamma$ , where  $ds$  denotes the arclength element. We refer to [12] and [13] for further details on moduli of curve families including the definitions of rectifiability and arclength in general metric spaces.

Given  $K \geq 1$  Ahlfors calls in [1] a homeomorphism of the plane (*geometrically*)  $K$ -*quasiconformal* if there is a constant  $K \geq 1$  such that for every family of curves  $\Gamma$  in  $\mathbb{R}^2$  the following inequalities hold

$$(2.8) \quad K^{-1} \text{mod}_2 f(\Gamma) \leq \text{mod}_2 \Gamma \leq K \text{mod}_2 f(\Gamma),$$

where  $f(\Gamma)$  denotes the image of the family  $\Gamma$  under  $f$ . A 1-quasiconformal mapping is actually conformal [2]. The same definition of geometric quasiconformality is usually given for mappings between Ahlfors regular spaces, but we will only need this in the case of  $\mathbb{R}^2$ .

**2.3.2. Modulus of measures.** The notion of the modulus of families of measures was defined and studied by Fuglede in [9]. In [11] the second author used Fuglede's modulus to study conformal dimension of various spaces. In the present paper we will use this notion in a similar way to study the distortion properties of quasisymmetric maps.

Let  $(X, \mu)$  be a measure space. Let  $\mathcal{L} = \{\lambda_i\}_{i \in I}$  be a collection of measures on  $X$  the domains of which contain the domain of  $\mu$ . A positive Borel function  $\rho : X \rightarrow [0, \infty)$  is said to be admissible for  $\mathcal{L}$  if  $\int_X \rho d\lambda_i \geq 1$  for every  $\lambda_i \in \mathcal{L}$ . The  $p$ -modulus of  $\mathcal{L}$  is

$$\text{mod}_p(\mathcal{L}) = \inf \int_X \rho^p d\mu,$$

where inf is over all  $\mathcal{L}$ -admissible functions  $\rho$ .

Given a family  $\mathcal{E} = \{E\}$  of Ahlfors  $d$ -regular subsets of  $\mathbb{R}^N$  and a measure  $\mu$  we will denote

$$\text{mod}_p(\mathcal{E}, \mu) := \text{mod}_p(\{\mathcal{H}_d|_E\}_{E \in \mathcal{E}}, \mu),$$

i.e. the modulus of the family of restrictions of  $\mathcal{H}_d$  to  $E \in \mathcal{E}$ .

If  $(X, \mu)$  is an Ahlfors  $D$ -regular metric space and  $\mathcal{E} = \{E\}$  is a family of Ahlfors  $d$ -regular sets in  $X$  then  $\text{mod}_{D/d}(\mathcal{E}, \mathcal{H}_D)$  will be called *the conformal modulus of  $\mathcal{E}$* . The motivation is that in  $\mathbb{R}^N$  conformal maps preserve the  $N$ -modulus of curves (see [2], [24]). Furthermore, we will say that a property holds for *almost every*  $E$  from  $\mathcal{E}$  if it fails only for a family  $\mathcal{E}_0 \subset \mathcal{E}$  such that  $\text{mod}_{D/d}(\mathcal{E}_0, \mathcal{H}_D) = 0$ .

Next, we summarize some of the properties of moduli that will be useful for us.

**Lemma 2.3.** *For every  $p > 1$  the following properties hold.*

- (1)  $\text{mod}_p \mathcal{L} \leq \text{mod}_p \mathcal{L}'$ , if  $\mathcal{L} \subset \mathcal{L}'$
- (2)  $\text{mod}_p \mathcal{L} \leq \sum_i \text{mod}_p \mathcal{L}_i$ , if  $\mathcal{L} = \bigcup_{i=1}^{\infty} \mathcal{L}_i$
- (3) (*Zierner's Lemma*) If  $1 < p < \infty$ ,  $\mathcal{L}_1 \subset \mathcal{L}_2 \subset \dots$  are families of complete measures and  $\mathcal{L} = \bigcup_{i=1}^{\infty} \mathcal{L}_i$  then  $\text{mod}_p \mathcal{L} = \lim_{i \rightarrow \infty} \text{mod}_p \mathcal{L}_i$ .
- (4) Suppose  $\mu(X) < \infty$  and  $t > t' \geq 1$ . If  $\text{mod}_{t'} \mathcal{L} > 0$  then  $\text{mod}_t \mathcal{L} > 0$ .

See [9] for (1) and (2). Property (3) is due to Zierner for families of continua in  $\mathbb{R}^N$ , see Lemma 2.3 in [25]. Recall here that a measure  $\mu$  is said to be complete if every subset of a zero measure set is also  $\mu$ -measurable.

The proof in the case of general measure families is the same as in [25]. The last property is an application of Hölder's inequality.

**2.3.3. Moduli of products.** Let  $(E, \lambda)$  and  $(Y, \nu)$  be two measure spaces with  $0 < \lambda(E) < \infty$ . Denote  $X = E \times Y$  and  $\mu = \lambda \times \nu$ . Let  $\mathcal{Y} = \{E \times \{y\} : y \in Y\}$  and  $\lambda_y(U \times \{y\}) = \lambda(U)$  for every  $\lambda$ -measurable  $U \subset E$ . Define

$$\mathcal{L} = \{\lambda_y : y \in Y\}.$$

**Lemma 2.4.** *With the notation as above for every  $p \geq 1$  we have*

$$(2.9) \quad \text{mod}_p \mathcal{L} = \frac{\mu(X)}{\lambda(E)^p} = \frac{\nu(Y)}{\lambda(E)^{p-1}}.$$

*Proof.* First note that since the function  $\rho(x, y) \equiv \lambda(E)^{-1}$  is admissible for  $\mathcal{L}$ , we have  $\text{mod}_p \mathcal{L} \leq \frac{\mu(X)}{\lambda(E)^p}$ .

To obtain the lower bound, note that for every  $\mathcal{L}$ -admissible  $\rho$  we have  $\int_E \rho(x, y) d\lambda \geq 1, \forall y \in Y$ , and therefore by Hölder's inequality we obtain that for every  $y \in Y$  the following holds

$$1 \leq \lambda(E)^{p-1} \int_E \rho^p(x, y) d\lambda.$$

Integrating both sides of this inequality with respect to  $\nu$  we obtain

$$\nu(Y) \leq \lambda(E)^{p-1} \int_X \rho^p(x, y) d\mu,$$

and therefore

$$\frac{\mu(X)}{\lambda(E)^p} = \frac{\nu(Y)\lambda(E)}{\lambda(E)^{p-1}} \leq \int_X \rho^p(x, y) d\mu.$$

Hence  $\text{mod}_p \mathcal{L} \geq \frac{\mu(X)}{\lambda(E)^p}$ . □

This proof is the same as in the classical case of curve families and is given here only for completeness purposes.

**2.4. Discrete modulus.** The following notion of discrete modulus of a family of subsets of  $X$  was introduced by the second author in [11]. It was inspired by an analogous notion for families of paths due to Heinonen and Koskela [14].

**Definition 2.5.** Let  $X$  be a metric space and  $\mathcal{E} = \{E\}$  be a collection of subsets of  $X$ . Let  $\mathcal{B} = \{B\}$  be a cover of  $X$  by balls and  $v : \mathcal{B} \rightarrow [0, \infty)$  a function. The pair  $(v, \mathcal{B})$  is *admissible for  $\mathcal{E}$*  if  $\frac{1}{5}B \cap \frac{1}{5}B' = \emptyset$ , whenever  $B \neq B'$ , and  $\sum_{\frac{1}{5}B \cap E \neq \emptyset} v(B) \geq 1$  for every  $E \in \mathcal{E}$ . We will write  $(v, \mathcal{B}) \wedge \mathcal{E}$ , whenever  $(v, \mathcal{B})$  is admissible for  $\mathcal{E}$ . Next, for  $\delta > 0$  we set

$$\text{d-mod}_p^\delta(\mathcal{E}) = \inf \left\{ \sum_{B \in \mathcal{B}} v(B)^p : (v, \mathcal{B}) \wedge \mathcal{E} \text{ and } \text{diam} B \leq \delta, \forall B \in \mathcal{B} \right\}.$$

Discrete  $p$ -modulus of  $\mathcal{E}$  is

$$\mathrm{d}\text{-mod}_p(\mathcal{E}) = \lim_{\delta \rightarrow 0} \mathrm{d}\text{-mod}_p^\delta(\mathcal{E}).$$

*Remark 2.6.* Note that we may assume that  $v(B_i) \leq 1, \forall i \in \mathbb{N}$ . Indeed, if  $(v, \mathcal{B})$  is an admissible pair for  $\mathcal{E}$  then so is  $(\tilde{v}, \mathcal{B})$ , where  $\tilde{v}(B_i) := \min(v(B_i), 1)$ . Therefore  $\mathrm{d}\text{-mod}_p(\mathcal{E}) \leq \mathrm{d}\text{-mod}_q(\mathcal{E})$  whenever  $p \geq q$  and it is only natural to introduce the following definition.

**Definition 2.7.** Let  $X$  be a metric space and  $\mathcal{E} = \{E\}$  be a collection of subsets of  $X$ . Define

$$\begin{aligned} \Delta(\mathcal{E}) &= \inf\{p : \mathrm{d}\text{-mod}_p(\mathcal{E}) = 0\} \\ &= \sup\{p : \mathrm{d}\text{-mod}_p(\mathcal{E}) > 0\}. \end{aligned}$$

The following lemma is a slight strengthening on Lemma 6.4 from [11].

**Lemma 2.8.** *Let  $t > 0$  and suppose  $\mathcal{E}$  is a collection of subsets in  $X$  such that  $\inf_{E \in \mathcal{E}} \mathcal{H}_t(E) > 0$ . Then for every  $q > 0$  we have*

$$(2.10) \quad \mathrm{d}\text{-mod}_q \mathcal{E} \leq \frac{\mathcal{H}_{qt}(X)}{(\inf_{E \in \mathcal{E}} \mathcal{H}_t(E))^q}.$$

*In particular if  $\mathcal{H}_D(X) < \infty$  for some  $D > 0$  then*

$$(2.11) \quad \mathrm{d}\text{-mod}_{D/t} \{E \in \mathcal{E} : \mathcal{H}_t(E) = \infty\} = 0.$$

Hence, for a family  $\mathcal{E}$  of subsets of  $X$  we always have

$$(2.12) \quad \Delta(\mathcal{E}) \leq \frac{\dim_H X}{\inf_{E \in \mathcal{E}} \dim E}.$$

In the proof of the lemma we will need the following well known result.

**Lemma 2.9 (Covering Lemma).** *Every family  $\mathcal{B}$  of balls of bounded diameter in a compact metric space  $X$  contains a countable subfamily of disjoint balls  $B_i \subset \mathcal{B}$  such that*

$$\bigcup_{B \in \mathcal{B}} B \subset \bigcup_i 5B_i.$$

*Proof of Lemma 2.8.* It is enough to show that for every  $\delta > 0$  we have

$$(2.13) \quad \mathrm{d}\text{-mod}_q^\delta \mathcal{E} \leq \frac{\mathcal{H}_{tq}^\delta(X)}{(\inf_{E \in \mathcal{E}} \mathcal{H}_t^\delta(E))^q}.$$

For that, suppose  $c > 0$  is chosen in such a way that  $H_t^\delta E \geq c > 0, \forall E \in \mathcal{E}$ . Next, by the covering lemma, for every  $0 < \varepsilon < \delta$  there is a covering  $\mathcal{B}$  of  $X$  by balls  $B_1, B_2, \dots$  with radii  $r_1, r_2, \dots$  such that  $\frac{1}{5}B_i \cap \frac{1}{5}B_j = \emptyset$ , for  $i \neq j$  and  $\sum_i r_i^{tq} < \mathcal{H}_{tq}^\delta(X) + \varepsilon$ . Note that if  $v(B_i) = r_i^t/c$  then  $(v, \mathcal{B})$  is admissible for  $\mathcal{E}$ . Indeed, since  $r_i < \delta$  it follows that for every  $E \in \mathcal{E}$  we have  $\sum_{B \cap E \neq \emptyset} v(B) \geq \frac{1}{c} H_t^\delta(E) \geq 1$ . Therefore,

$$\sum_i v(B_i)^q = \frac{1}{c^q} \sum_i r_i^{tq} < \frac{1}{c^q} (\mathcal{H}_{tq}^\delta(X) + \varepsilon).$$



Taking  $\varepsilon \downarrow 0$  and then  $c \uparrow \inf_{E \in \mathcal{E}} \mathcal{H}_t^\delta(E)$  we obtain the required equality for  $\delta > 0$ .  $\square$

Note that in the case of a family  $\mathcal{E}$  of  $d$ -regular subsets of  $X$  Lemma 2.8 implies

$$\text{d-mod}_{D/d} \mathcal{E} \leq \frac{\mathcal{H}_D(X)}{(\inf_{E \in \mathcal{E}} \mathcal{H}_d(E))^{D/d}}.$$

**2.5. Bojarski's Lemma.** The next well known result is due to Bojarski [7], and will be used in the proof of Theorem 3.3 below. It is a consequence of the boundedness of the Hardy-Littlewood maximal operator on doubling metric measure spaces (see also [12]). The proof of this more general version is the same as in [7].

**Lemma 2.10** (Bojarski Lemma). *Suppose  $\mathcal{B} = \{B_i, B_2, \dots\}$  is a countable collection of balls in a doubling metric measure space  $(X, \mu)$  and  $a_i \geq 0$  are real numbers. Then there is a positive constant  $C$  such that*

$$(2.14) \quad \int_X \left( \sum_{\mathcal{B}} a_i \chi_{AB_i}(x) \right)^q d\mu \leq C(A, p, \mu) \int_X \left( \sum_{\mathcal{B}} a_i \chi_{B_i}(x) \right)^q d\mu$$

for every  $1 < q < \infty$  and  $A > 1$ .

### 3. STATEMENT OF RESULTS

**3.1. Modulus and quasisymmetries.** It is well known that quasiconformal mappings of  $\mathbb{R}^N$ ,  $N \geq 2$ , are absolutely continuous on almost all rectifiable curves, see e.g [24]. Here “almost all” means that the “exceptional” family of curves on which  $f$  is not absolutely continuous has zero conformal modulus. Therefore, the collection of curves  $\gamma$  in  $\mathbb{R}^N$  such that  $f(\gamma)$  is unrectifiable also has zero modulus:

$$(3.1) \quad \text{mod}_N(\{\gamma : \mathcal{H}_1(f(\gamma)) = \infty\}, \mathcal{H}_N) = 0.$$

Given a metric space  $X$  let  $\mathcal{A}_d(X)$  be the collection of bounded Ahlfors  $d$ -regular subsets in  $X$ .

**Theorem 3.1.** *Suppose  $D = \dim X$ ,  $D' = \dim Y$  and  $Y$  is Ahlfors  $D'$ -regular. Then for every quasisymmetric mapping  $f : X \rightarrow Y$  the following holds*

$$(3.2) \quad \text{mod}_{\frac{D'}{d'}}(\{E \in \mathcal{A}_d(X) : \mathcal{H}_{d'}(f(E)) = \infty\}, \mathcal{H}_{\frac{d}{d'} D'}) = 0.$$

In particular, if  $D/d = D'/d'$  then

$$(3.3) \quad \text{mod}_{\frac{D}{d}}(\{E \in \mathcal{A}_d(X) : \mathcal{H}_{\frac{D'}{d'}}(f(E)) = \infty\}, \mathcal{H}_D) = 0.$$

Using the terminology of Section 2, this theorem clearly implies the following.

**Corollary 3.2.** *For every quasisymmetric map  $f$  of the metric space  $X$  into an Ahlfors regular space and a.e. bounded Ahlfors  $d$ -regular set  $E \subset X$*

$$(3.4) \quad \frac{\dim f(E)}{\dim E} \leq \frac{\dim f(X)}{\dim X}.$$

*In particular, if  $\dim f(X) \leq \dim X$  then for a.e. bounded  $d$ -regular  $E \subset X$*

$$(3.5) \quad \dim f(E) \leq \dim E.$$

Theorem 3.1 follows from the following theorem which is the main result of the paper.

**Theorem 3.3.** *Let  $D > 1$ ,  $d > 0$ . Suppose  $\mu$  is an upper  $D$ -regular measure on  $X$ . If there is a lower  $d$ -regular family of complete measures  $\mathcal{L} = \{\lambda_E\}_{E \in \mathcal{E}}$  on  $X$  such that  $\text{mod}_{\frac{D}{d}}(\mathcal{L}, \mu) > 0$  then for every quasisymmetric mapping  $f : X \rightarrow Y$  the following holds*

$$(3.6) \quad \frac{D}{d} \leq \frac{\dim_H f(X)}{\inf_{E \in \mathcal{E}} \dim_H f(E)}.$$

*Furthermore, if every  $E \in \mathcal{E}$  is bounded in  $X$  and for every bounded set  $U \subset Y$  we have  $\mathcal{H}_{D'}(U) < \infty$  then*

$$(3.7) \quad \text{mod}_{\frac{D}{d}}(\{\lambda_E \in \mathcal{L} : \mathcal{H}_{d'}(f(E)) = \infty\}, \mu) = 0,$$

*whenever  $\mu$  is upper  $D$ -regular, with  $D \geq \frac{d}{d'} \cdot D'$ .*

**3.2. Exceptional “vertical” translates in  $\mathbb{R}^N$ .** An important consequence of (3.1) is that for a QC map of  $\mathbb{R}^N$

$$(3.8) \quad \mathcal{H}_{N-1}(\{y \in \mathbb{R}^\perp : \dim f([0, 1] + y) > 1\}) = 0.$$

In [5] Balogh, Monti and Tyson greatly generalized equation (3.8) to Sobolev mappings from  $\mathbb{R}^N$  into a separable metric space  $Y$ . In particular they showed that if  $f$  is a quasiconformal mapping of  $\mathbb{R}^N$ ,  $1 \leq n \leq N$  and  $d' > n$  then

$$(3.9) \quad \mathcal{H}_{\frac{n}{d'}N-n}\{y \in (\mathbb{R}^n)^\perp : \dim f(\mathbb{R}^n + y) > d'\} = 0.$$

Theorem 3.3 may be used in a similar way.

**Theorem 3.4.** *Let  $E \subset \mathbb{R}^n$  be a closed, Ahlfors  $d$ -regular set. Then for every quasisymmetric map  $f : \mathbb{R}^N \rightarrow Y$ , with  $\dim_H Y = D'$  and  $d' \in (0, D')$  we have*

$$(3.10) \quad \mathcal{H}_{\frac{d}{d'}D'-d}(\{y \in (\mathbb{R}^n)^\perp : \dim_H f(E + y) > d'\}) = 0.$$

*In particular,*

$$\dim_H \{y \in (\mathbb{R}^n)^\perp : \dim_H f(E + y) > d'\} \leq \min\left(\frac{d}{d'}D' - d, N - n\right).$$

Note, that for  $E = \mathbb{R}^n$  and  $Y = \mathbb{R}^N$ , and hence  $d = n, D = N$ , we recover (3.9). In [5] the authors asked if it is possible generalize (3.9) to more general source spaces than  $\mathbb{R}^N$ , see Problem 6.5 of [5]. Theorems 3.1 and 3.3 may be thought of as results in that direction, since they imply Theorem 3.4 (and in particular equation (3.9)).

For a given  $E$  and  $d' > \dim E$  Theorem 3.4 estimates the size of those  $y$ 's such that  $\dim f(E + y) > d'$ . Suppose now that a set  $Y \subset (\mathbb{R}^N)^\perp$  is given. How large can we make  $\dim f(E + y)$  for all  $y$ 's from  $Y$  simultaneously? This is answered by rewriting inequality (1.1) of Theorem 1.1 as follows.

$$d' := \inf_{y \in Y} \dim_H f(E \times \{y\}) \leq \frac{\dim_H f(E \times Y)}{\dim_H(E \times Y)} \cdot \dim_H E.$$

The first part of Theorem 1.2 is a particular case of this for the quasiconformal mappings of the plane.

*Remark 3.5.* The theorems above actually hold for more general sets than just Ahlfors regular ones. Namely, we can assume that  $E$  satisfies a weaker condition

$$(3.11) \quad r^{d+\varepsilon} \lesssim \lambda_E(E \cap B(x, r)) \lesssim r^{d-\varepsilon}, \quad \forall x \in E, \forall r \in (0, r_0).$$

The proofs of the corresponding results are very similar to the ones in [11] and will not be given in the present paper.

**3.3. Increasing dimension of many curves and sets.** It is known that there are quasiconformal maps of  $\mathbb{R}^N$  such that  $f(\mathbb{R} + y)$  has infinite length for a large set of  $y$ 's from  $\mathbb{R}^\perp$ , see [15], [17]. Even though in all these examples the images  $f(\mathbb{R} + y)$  have infinite length they contain many rectifiable arcs. So it is natural to ask if there is an uncountable set  $E \subset [0, 1]$  and a quasiconformal map  $f$  of  $\mathbb{R}^2$  so that  $f(\mathbb{R} + iy)$  contains no rectifiable subarc for any  $y \in E$ ? Theorem 1.3 shows that this indeed is the case. In fact  $E$  may be taken as large as we want as long as it does not have positive length.

Can a quasiconformal map increase the dimension of every subarc for an uncountable collection of lines? This question appears as Problem 6.3 in [5] and Question 5.3 in [8].

**Theorem 3.6.** *For each  $0 < d < 1$  there is a Cantor set  $E \subset i\mathbb{R}$  of dimension  $d$  such that for any  $\epsilon > 0$  there is a quasiconformal map  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that for every interval  $(a, b) \subset \mathbb{R}$*

$$\inf_{y \in E} \dim_H f((a, b) + y) \geq \frac{2}{1+d} - \epsilon,$$

and

$$\inf_{0 \leq x \leq 1} \dim_H f(x + iE) \geq \frac{2d}{d+1} - \epsilon,$$

where  $x + iE = \{x + iy : y \in E\}$ . Moreover, for any subset  $F \subset E$ , we have

$$\inf_{0 \leq x \leq 1} \dim_H (f(x + iF)) \geq \frac{2\dim_H(F)}{d+1} - \epsilon.$$

The proof of Theorem 3.6 uses conformal mappings and does not generalize to  $N > 2$ . Consequently, we do not know if it is possible to increase the dimension of every subarc of  $\mathbb{R} + y$  for  $y$  in a large set of parameters. It is more feasible to expect that Hausdorff dimension may increase. We do not deal with such higher dimensional constructions in the present paper.

The following corollary shows the sharpness of (3.9) for  $N = 2$  and  $n = 1$ .

**Corollary 3.7.** *For every  $\delta > 1$  and  $\varepsilon > 0$  there is a quasiconformal map  $f$  of the plane such that for every interval  $(a, b) \subset \mathbb{R}$  we have*

$$(3.12) \quad \dim_H \{y \in \mathbb{R} : \dim_H f((a, b) + iy) \geq \delta\} \geq \frac{2}{\delta} - 1 - \varepsilon.$$

*Proof.* Theorem 3.6 implies that for every  $0 < d < 1$  and  $\varepsilon > 0$  there is a quasiconformal mapping  $f$  of  $\mathbb{R}^2$  such that

$$\dim_H \{y \in i\mathbb{R} : \dim f(\mathbb{R} + y) > \frac{2}{1+d} - \frac{\varepsilon}{2}\} \geq \dim_H E = d.$$

Here  $\varepsilon > 0$  is chosen so small that  $\delta = \frac{2}{1+d} - \frac{\varepsilon}{2} > 1$ . Now,

$$d = \frac{2}{\delta + (\varepsilon/2)} - 1 \geq \frac{2}{\delta} \left(1 - \frac{\varepsilon}{2\delta}\right) - 1 \geq \frac{2}{\delta} - 1 - \varepsilon,$$

and therefore

$$\dim_H \{y \in i\mathbb{R} : \dim f(\mathbb{R} + y) > \delta\} \geq \frac{2}{\delta} - 1 - \varepsilon. \quad \square$$

#### 4. PROOFS OF THEOREM 3.1 AND THEOREM 3.3

One of the main ingredients of the proof of Theorem 3.3 is the following result, which states that if  $\mathcal{E}$  is a “nice” family of subsets of  $X$  then  $f$  does not distort the modulus of  $\mathcal{E}$  too much. Lemma 4.1 is very similar to Lemma 6.5 of [11], the main difference being the range of  $q$ ’s for which it holds. In [11] only the case of  $d = 1$  was considered.

**Lemma 4.1.** *Let  $D > 1$  and  $d > 0$ . Suppose  $(X, \mu)$  is a doubling metric measure space, such that  $\mu$  is upper  $D$ -regular. Then for every lower  $d$ -regular family of measures  $\mathcal{L} = \{\lambda_E\}_{E \in \mathcal{E}}$  and every quasisymmetric mapping  $f : X \rightarrow Y$  there is a finite constant  $C$ , such that*

$$(4.1) \quad \text{mod}_{\frac{D}{d}} \mathcal{L}_{\mathcal{E}} \leq C d \cdot \text{mod}_{\frac{D}{d}} f(\mathcal{E}).$$

Here  $C$  depends on  $\eta$ , and the constants from (2.2) and (2.3).

*Proof.* We will show that for every  $\delta > 0$  the following inequality holds

$$\text{mod}_{\frac{D}{d}} \mathcal{L} \leq C d \cdot \text{mod}_{\frac{D}{d}}^{\delta} f(\mathcal{E}),$$

with  $C$  independent of  $\delta$ . Taking  $\delta \rightarrow 0$  we will arrive to the desired inequality (4.1). To avoid cumbersome notation, below we denote  $q = \frac{D}{d}$ .

Suppose  $(v, \mathcal{B}')$  is an  $f(\mathcal{E})$ -admissible pair, where  $\mathcal{B}' = \{B'_i\}_{i=1}^\infty$  is a cover of  $f(X)$  by balls  $B'_i$  of radii  $r'_i < \delta$ . Since  $f$  is quasisymmetric, we may choose  $B_i \subset X$  with radii  $r_i$  so that

$$(4.2) \quad \frac{1}{H}B_i \subset f^{-1}\left(\frac{1}{5}B'_i\right) \subset f^{-1}(B'_i) \subset B_i.$$

Note that since  $\mathcal{B}'$  is admissible it follows that  $\frac{1}{H}B_i \cap \frac{1}{H}B_j = \emptyset$  whenever  $i \neq j$ .

We want to construct an  $\mathcal{L}$ -admissible function  $\rho$  such that  $\int_X \rho^q d\mu \leq C \sum_{B \in \mathcal{B}'} v(B)^q$ . Let  $a_i = \frac{v(B'_i)}{r_i^d}$  and define

$$\rho(x) = \sum_i a_i \chi_{2B_i}(x).$$

Then for every  $E \in \mathcal{E}$  the following holds

$$\begin{aligned} \int_E \rho d\lambda_E &= \int_E \sum_i a_i \chi_{2B_i}(x) d\lambda_E \geq \int_E \sum_{i: B_i \cap E \neq \emptyset} a_i \chi_{2B_i}(x) d\lambda_E \\ &= \sum_{i: B_i \cap E \neq \emptyset} a_i \int_{E \cap 2B_i} d\lambda_E = \sum_{i: B_i \cap E \neq \emptyset} a_i \lambda_E(E \cap 2B_i). \end{aligned}$$

Now, since  $B_i \cap E \neq \emptyset$  there is an  $x \in E$  such that  $B(x, r_i) \subset 2B_i$ . Therefore,

$$\lambda_E(E \cap 2B_i) \geq \lambda_E(B(x, r_i)) \geq C_1^{-1} r_i^d.$$

Therefore

$$\int_E \rho d\lambda_E \geq C_1^{-1} \sum_{i: f(E) \cap B'_i \neq \emptyset} v(B'_i) \geq C_1^{-1}.$$

It follows that  $\text{mod}_q(\mathcal{L}) \leq C_1^q \int_X \rho^q d\mu$ . To estimate  $\int_X \rho^q d\mu$  we use Bojarski's lemma to obtain

$$\int_X \rho^q d\mu = \int_X \left( \sum_i a_i \chi_{2B_i}(x) \right)^q d\mu \leq C(2H, q, \mu) \int_X \left( \sum_i a_i \chi_{\frac{1}{H}B_i}(x) \right)^q d\mu.$$

Since  $\frac{1}{H}B_i$ -s are disjoint, it follows from (2.2) that

$$(4.3) \quad \int_X \rho^q d\mu \leq C(2H, q, \mu) \sum_i a_i^q \mu\left(\frac{1}{H}B_i\right) \lesssim \sum_i v(B'_i)^q r_i^{D-qd} = \sum_i v(B'_i)^q,$$

since  $D - qd = 0$ . Taking infimum over all  $f(\mathcal{E})$ -admissible pairs  $(v', \mathcal{B}')$  we obtain  $\text{mod}_q \mathcal{L} \leq C \text{d-mod}_q^\delta f(\mathcal{E})$  for some  $C$  which is independent of  $\delta$ . As noted above this completes the proof.  $\square$

*Remark 4.2.* If we also assume that  $X$  is a locally compact space then we could conclude that inequality

$$\text{mod}_q \mathcal{L} \leq C \text{d-mod}_q^\delta f(\mathcal{E}),$$

holds for every  $1 < q \leq D/d$ . Indeed, it is easy to see that under this additional assumption we may choose our coverings  $\mathcal{B}'$  and  $\mathcal{B}$  so that  $r_i < 1, \forall i \in \mathbb{N}$ . Hence, since  $D - qd > 0$ , the inequality (4.3) would hold for all such values of  $q$ . We will not this stronger result below and mentioned it here only for the convenience of the reader.

**Corollary 4.3.** *Under the conditions of Lemma 4.1 assume that  $\text{mod}_q \mathcal{L} > 0$ , for some  $1 \leq q \leq D/d$ . Then for every quasisymmetric map  $f$  the following holds*

$$(4.4) \quad \frac{D}{d} \leq \Delta(f\mathcal{E}).$$

*Proof.* By Lemma 2.3 we have  $\text{mod}_{D/d} \mathcal{L} > 0$ . Therefore by (4.1) we get

$$\text{d-mod}_{D/d}(f\mathcal{E}) \gtrsim \text{mod}_{D/d} \mathcal{L} > 0,$$

which clearly implies (4.4).  $\square$

*Proof of Theorem 3.3.* The first part of the theorem is essentially obtained by combining inequalities (4.4) and (2.12). To be more precise, fix  $t < \inf_{E \in \mathcal{E}} \dim f(E)$ . Then  $\mathcal{H}_t(f(E)) = \lim_{\delta \rightarrow 0} \mathcal{H}_t^\delta(f(E)) = \infty$  for all  $E \in \mathcal{E}$ . Let  $\mathcal{E}_i := \{E \in \mathcal{E} : \mathcal{H}_t^{1/i}(f(E)) \geq 1\}$ . From our assumption it follows that  $\bigcup_{i=1}^\infty \mathcal{E}_i = \mathcal{E}$ . Then from Lemma 2.8 it follows that

$$\Delta(f(\mathcal{E}_i)) \leq \frac{\dim_H f(X)}{t}$$

for every  $i \in \mathbb{N}$ . Now, if  $\mathcal{L}_i = \{\lambda_E \in \mathcal{L} : E \in \mathcal{E}_i\}$  then  $\bigcup_{i=1}^\infty \mathcal{L}_i = \mathcal{L}$ . Since  $\text{mod}_{D/d} \mathcal{L} > 0$  it follows from Ziemer's Lemma that there is an  $i_0 \in \mathbb{N}$  such that for  $i > i_0$  we have  $\text{mod}_{D/d} \mathcal{L}_i > 0$  and hence by (4.4) also  $\frac{D}{d} \leq \Delta(f\mathcal{E}_i)$ . Combining the obtained upper and lower bounds for  $\Delta(f\mathcal{E}_i)$  we obtain  $\frac{D}{d} \leq \frac{\dim_H f(X)}{t}$ . Since  $t < \inf_{E \in \mathcal{E}} \dim f(E)$  is arbitrary we obtain the desired inequality (3.6).

To prove equality (3.7) let  $\mathcal{L}(d') = \{\lambda \in \mathcal{L} : \mathcal{H}_{d'}(f(E)) = \infty\}$  and

$$\mathcal{E}'_i = \{E \in \mathcal{E} : \mathcal{H}_{d'}(f(E)) = \infty, f(E) \subset B(y, i)\},$$

where  $y \in Y$  is any fixed point in  $Y$ . Furthermore, if  $\mathcal{L}'_i = \{\lambda_E\}_{E \in \mathcal{E}'_i}$  then  $\mathcal{L}'_i \subset \mathcal{L}'_{i+1}$  and since quasisymmetric mappings take bounded sets in  $X$  to bounded sets in  $Y$  (see [12]) we obtain  $\mathcal{L}(d') = \bigcup_{i=1}^\infty \mathcal{L}'_i$ .

Now, by Lemma 2.8  $\text{d-mod}_{D/d} f(\mathcal{E}'_i) = 0$  for every  $i \geq 1$ . Therefore by Lemma 4.1 we have  $\text{mod}_{D/d}(\mathcal{L}'_i, \mu) = 0, i \geq 1$ . Hence by Ziemer's lemma  $\text{mod}_{D/d} \mathcal{L}(d') = 0$  and the proof is complete.  $\square$

*Proof of Theorem 3.1.* Let  $\mathcal{A}_d^N(X)$  be the collection of bounded Ahlfors  $d$ -regular subsets of  $X$  such that for every  $E \in \mathcal{A}_d^N(X)$  and  $x \in E$

$$\frac{1}{N} r^d \leq \mathcal{H}_d(E \cap B(x, r)) \leq N r^d.$$

Clearly, then  $\mathcal{A}_d(X) = \bigcup_{N=1}^{\infty} \mathcal{A}_d^N(X)$  and

$$\{E \in \mathcal{A}_d(X) : \mathcal{H}_{d'}(f(E)) = \infty\} = \bigcup_{N=1}^{\infty} \{E \in \mathcal{A}_d^N(X) : \mathcal{H}_{d'}(f(E)) = \infty\}.$$

Since  $Y$  is Ahlfors  $D'$ -regular we have that  $\mathcal{H}_{D'}(U) < \infty$  for every bounded subset of  $Y$ . Therefore, by the second part of Theorem 3.3 we have for every  $N \geq 1$  the following

$$\text{mod}_{\frac{D'}{d}}(\{E \in \mathcal{A}_d^N(X) : \mathcal{H}_{d'}(f(E)) = \infty\}, \mathcal{H}_{\frac{d}{d'}D'}) = 0,$$

and by Ziemer's Lemma we obtain inequality (3.2).  $\square$

## 5. SIZE OF THE SET OF EXCEPTIONAL TRANSLATES

**5.1. Proof of Theorem 3.4.** We will need the following result in order to be able to apply Frostman's lemma. The proof is almost the same as the one for Lemma 3.1 in [5] (the only difference being the space to which we apply the Tube lemma) and we present it here only for reader's convenience.

**Lemma 5.1.** *Let  $f : \mathbb{R}^N \rightarrow Y$  be a homeomorphism and  $E \subset \mathbb{R}^n$  be a closed set. Then for each  $\alpha \in [d, N)$  the set*

$$\mathcal{E}_f(E, \alpha) = \{y \in (\mathbb{R}^n)^\perp : \mathcal{H}_\alpha^\infty f(E + y) > 0\}$$

*is a Borel set.*

*Proof.* Exhausting  $\mathbb{R}^N$  with an increasing sequence of compact sets  $\{K_i\}$  and letting

$$\mathcal{E}(\alpha, i, \delta) = \{a \in (\mathbb{R}^n)^\perp : \mathcal{H}_\alpha^\infty f((E + y) \cap K_i) > \delta\},$$

it is enough to show that  $E(\alpha, i, \delta)$  is a closed set, since

$$\mathcal{E}_f(E, \alpha) = \bigcup_i \bigcup_{\delta > 0} E(\alpha, i, \delta).$$

Let  $a_i \in E(\alpha, i, \delta)$  and  $a_i \rightarrow a$ . For a covering of  $f((E + a) \cap K_i)$  by balls  $\{A_k\}$  of radii  $r_k$  denote  $B_k = f^{-1}(A_k)$ . Since  $f$  is continuous,  $E$  is closed and hence  $(E + a) \cap K_i$  is compact it follows that there is a neighborhood  $U$  of  $a$  in  $(\mathbb{R}^n)^\perp$  such that  $(E \times U) \cap K_i \subset \bigcup_k B_k$ . Indeed, this follows from the Tube lemma which we can apply to  $E \times \mathbb{R}^{N-n}$  since  $E$  is closed and therefore  $E \cap K_i$  is compact, see [20]. Taking  $j$  large enough we will have  $a_j \in U$  and therefore  $f((E + a_j) \cap K_i) \subset \bigcup_k A_k$  and therefore  $\sum_k r_k > \delta$ . Thus  $a \in E(\alpha, i, \delta)$  as well and  $E(\alpha, i, \delta)$  is closed.  $\square$

*Proof of Theorem 3.4.* Fix a point  $x \in X$  and let  $E_i = E \cap B(x, i)$ . Note that  $E_i$  is also  $d$ -regular. Denote,

$$\begin{aligned} Y(d') &= \{y \in \mathbb{R}^{N-n} : \dim f(E \times \{y\}) > d'\}, \\ Y_i(d') &= \{y \in \mathbb{R}^{N-n} : \dim f(E_i \times \{y\}) > d'\}. \end{aligned}$$

Then, since  $\dim f(E \times \{y\}) = \lim_{i \rightarrow \infty} \dim f(E_i \times \{y\})$ , we have

$$Y(d') = \bigcup_{i=1}^{\infty} Y_i(d').$$

Denoting  $\beta = \dim_H Y(d')$  and  $\beta_i = \dim_H Y_i(d')$  we will therefore have  $\beta = \lim_{i \rightarrow \infty} \beta_i$ .

Now, fix an integer  $i \in \mathbb{N}$ . Since  $Y_i(d')$  is a Borel set, by Frostman's Lemma for every  $t < \beta_i$  there is a nontrivial measure  $\nu$  on  $Y_i(d')$  satisfying  $\nu(B(y, r)) < Cr^t$ . Now let  $X = E_i \times Y_i(d')$  and  $\mu = \lambda \times \nu$ , where  $\lambda$  is the Ahlfors  $d$ -regular measure on  $E$ . Then  $\mu(B(x, r)) \lesssim r^{d+t}$  for all  $x \in X$  and  $0 < r < \text{diam} X$ . Therefore letting

$$\mathcal{Y} = \mathcal{Y}_i(d') = \{E_i \times \{y\} : y \in Y_i(d')\},$$

Lemmas 2.4 and 4.1 imply that the following inequalities hold,

$$0 < \frac{\mu(X)}{\lambda(E_i)^p} = \text{mod}_{\frac{d+t}{d}} \mathcal{L} \lesssim \text{d-mod}_{\frac{d+t}{d}} f(\mathcal{Y}),$$

where the family  $\mathcal{L}$  is defined as in Lemma 2.4. Combining with Lemma 2.8 we obtain

$$(5.1) \quad \frac{d+t}{d} \leq \Delta f(\mathcal{Y}) \leq \frac{D'}{d'}.$$

Since  $t < \beta_i$  can be taken as close to  $\beta_i$  as we want it follows that

$$\frac{d + \beta_i}{d} \leq \frac{D'}{d'}.$$

Therefore  $\dim Y_i(d') = \beta_i \leq \frac{d}{d'} D' - d$ , and taking  $i \rightarrow \infty$  we obtain

$$\dim Y(d') = \beta \leq \frac{d}{d'} D' - d.$$

Now, note that  $Y(d') = \cup_{i=1}^{\infty} Y(d' + 1/i)$  and since

$$\dim Y(d' + 1/i) \leq \frac{d}{d' + 1/i} D' - d < \frac{d}{d'} D' - d,$$

we have  $\mathcal{H}_{\frac{d}{d'} D' - d}(Y(d' + 1/i)) = 0$  for every  $i \geq 1$ . Therefore  $\mathcal{H}_{\frac{d}{d'} D' - d} Y(d') = 0$ .  $\square$

*Proof of Theorem 1.1.* Since  $E$  is Ahlfors  $d$ -regular the packing dimension of  $E$  is equal to its Hausdorff dimension, see Theorem 6.13 of [19]. Therefore we also have

$$\dim_H(E \times Y) = \dim_H E + \dim_H Y = d + \beta,$$

see e.g. Corollary 8.11 of [19].

Hence, inequality (5.1) in the proof of Theorem 3.4 implies that for every compact (and therefore also closed)  $d$ -regular set  $E$  and a Borel set  $Y$  we have for  $\mathcal{Y} = \{E \times \{y\} : y \in Y\}$  the inequality

$$\frac{\dim(E \times Y)}{\dim E} \leq \Delta f(\mathcal{Y}) \leq \frac{\dim f(X)}{d'},$$



provided  $\dim f(E \times \{y\}) \geq d', \forall y \in Y$ .

Taking  $d' = \inf_{y \in Y} \dim f(E \times \{y\})$  we obtain inequality (1.1).  $\square$

## 6. PROOF OF THEOREM 1.3

The idea is quite simple. We start by mapping horizontal tubes to nearly horizontal tubes that oscillate. Inside these, we build thinner tubes that oscillate on a smaller scale, and continue by induction, obtaining in the limit a Cantor set of curves each of which oscillates at infinitely many scales, hence has no rectifiable subarc. Using sufficiently many sufficiently thin tubes we can keep the quasiconstant bounded while making the Hausdorff measure as large as we want.

The proof is essentially a sequence of pictures. First choose a diffeomorphism  $\varphi$  of the unit square  $Q = [0, 1]^2$  to itself that is the identity on the boundary, and translates the vertical segment  $V = \{\frac{1}{2}\} \times [\frac{1}{4}, \frac{3}{4}]$  up by  $1/8$  to the segment  $\{\frac{1}{2}\} \times [\frac{3}{8}, \frac{7}{8}]$ . Thus segments of the form  $S_y = [0, 1] \times \{y\} \subset Q$  have curved images with the same endpoints as  $S_y$ , but deviate by at least  $\frac{1}{8}$  from  $S_y$  for  $y \in [\frac{1}{4}, \frac{3}{4}]$ . See Figure 1.

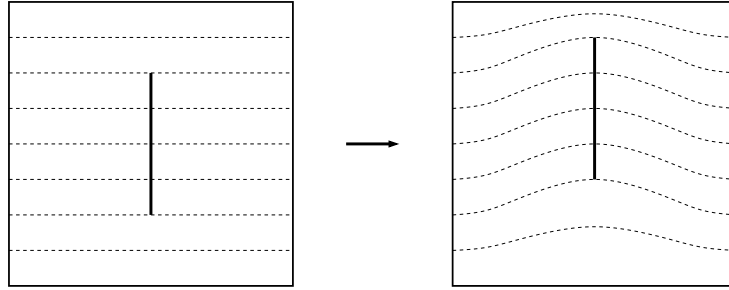


FIGURE 1. Choose a smooth self-map of  $Q$  that is the identity on  $\partial Q$ , but causes horizontal segments near the middle of  $Q$  to bend by a definite amount.

Let  $\gamma = \varphi(S_y)$  for  $y \in [\frac{1}{4}, \frac{3}{4}]$ . Choose a large integer  $n$  and divide  $\gamma$  into segments with endpoints  $z_k = (x_k, y_k)$  where  $x = \frac{k}{n}$ ,  $k = 0, 1, \dots, n$ . Let  $\gamma_n$  be the polygonal path with these vertices. At each  $z_k$ ,  $k = 1, \dots, n-1$ , draw a segment perpendicular to and above  $\gamma$  whose length is the same as the  $(k-1)$ st segment in  $\gamma_n$ . The other endpoint is denoted  $w_k$ . Let  $\tilde{\gamma}$  be the polygonal curve with vertices  $w_0, \dots, w_n$ . See Figure 2.

The reason we defined  $\tilde{\gamma}_n$  as we did (and did not simply translate  $\gamma_n$  upwards), is so that the region between the curves can be divided into quadrilaterals that are very nearly squares. We will denote this region  $T_n$  and call it a “tube”. Since  $\gamma$  is a smooth curve, adjacent segments of  $\gamma_n$  have angles that agree to within  $O(1/n)$ ; thus adjacent perpendicular segments have angles that agree to within  $O(1/n)$ . Thus each of the quadrilaterals formed by  $\gamma_n, \tilde{\gamma}_n$  and the segments joining their vertices have all angles within  $O(1/n)$  of 90 degrees. See Figure 3.

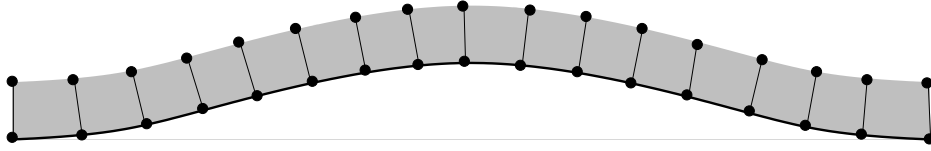


FIGURE 2. We inscribe a polygonal curve  $\gamma_n$  in  $\gamma$  and add perpendicular segments. We then connect the new vertices to form a path  $\tilde{\gamma}_n$  that is almost parallel to  $\gamma_n$ .

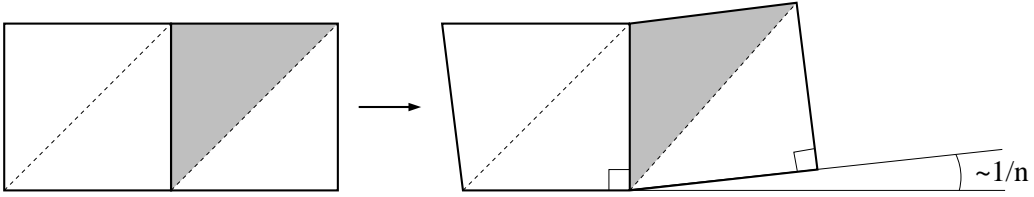


FIGURE 3. Two adjacent squares are mapped to adjacent quadrilaterals that are almost squares (with error  $O(1/n)$ ).

By adding diagonals and using the unique piecewise linear map between corresponding triangles, we get a quasiconformal map from a true square to each of our quadrilaterals with dilatation bounded by  $O(1/n)$ . Piecing these together gives us a map from a  $1 \times \frac{1}{n}$  rectangle to our tube  $T_n$ . See Figure 4.

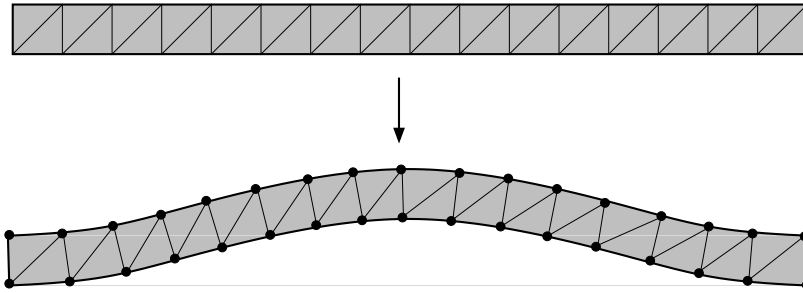


FIGURE 4. Piecewise linear maps define a  $1 + O(\frac{1}{n})$ -QC map from a rectangle to a tube.

By repeating the construction we can map several parallel straight tubes to several curved tubes as in Figure 5. We assume there are  $\simeq n$  tubes, all have width  $\simeq 1/n$  and are vertically separated by  $\simeq 1/n$ . This, combined with the fact that the edges of the tubes have bounded slope implies the piecewise linear map previously defined on the union of tubes can be extended to a quasiconformal map  $f_n$  of  $Q$  to itself with uniformly bounded constant (independent of  $n$ ).

Now we must repeat the construction at smaller scales, without letting the dilatations blow up. However, this is quite simple. Suppose we have

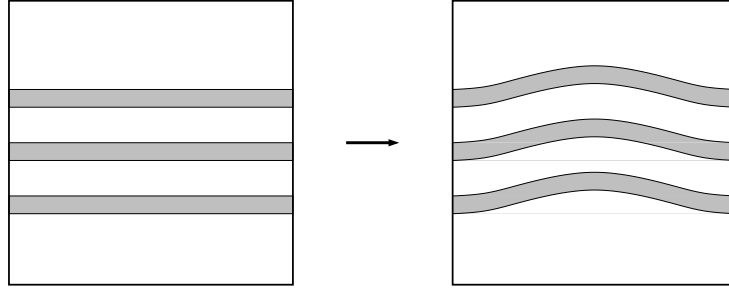


FIGURE 5. Repeat the construction on several parallel tubes

constructed a quasiconformal map  $g_k : Q \rightarrow Q$  and nested families  $\mathcal{I}_1 \supset \mathcal{I}_2 \supset \dots$  of intervals on the left side of  $Q$ . Each interval in  $\mathcal{I}_k$  has equal length and is both the left side of a rectangle and the left side of a tube that is the image of this rectangle under  $g_k$ .

For any  $I \in \mathcal{I}_k$ , define a subcollection of  $\simeq n_{k+1}$  sub-intervals in the middle half of  $I$  that have length  $|I|/n_{k+1}$  and are separated from each other by at least  $|I|/n_{k+1}$ . Doing this for each  $I \in \mathcal{I}_k$  defines  $\mathcal{I}_{k+1}$ .

For each  $J \in \mathcal{I}_{k+1}$ , divide the rectangle with vertical side  $J$  and horizontal side of length 1 into squares. For each such square, define the map  $f_{n_{k+1}}$  as before. This defines a  $O(1 + n_{k+1}^{-1})$ -quasiconformal map from the  $(k+1)$ st generation rectangles to the  $(k+1)$ st generation tubes. Also as before, the map can be extended to a map of the  $k$ th generation rectangles to the  $k$ th generation tubes with uniformly bounded dilatation. Outside the  $k$ th generation rectangles, we let it be the identity. This defines  $f_{n_{k+1}}$  on  $Q$ .

Let  $g_{k+1} = g_k \circ f_{n_{k+1}}$ . Inside the  $(k+1)$ st generation tubes the dilatation of  $g_{k+1}$  is bounded by

$$O\left(\frac{1}{n_1} + \dots + \frac{1}{n_{k+1}}\right),$$

and outside the  $k$ th generation tubes, the map is unchanged, so the dilatation is bounded by

$$O\left(\frac{1}{n_1} + \dots + \frac{1}{n_k}\right).$$

Between the  $k$ th and  $(k+1)$ st generation tubes  $g_{k+1}$  and  $g_k$  differ by a single map with bounded distortion, so the dilatation is still bounded by  $O(\frac{1}{n_1} + \dots + \frac{1}{n_k})$ , (but with possibly bigger constant).

Thus if we choose  $n_k$  to increase so quickly that  $\sum_k n_k^{-1}$  converges to a small value, the dilatations will be uniformly bounded. In this case, we can take a limit and obtain a map and a Cantor set  $E$  so that every segment  $S_y$ ,  $y \in E$  has a definite oscillation near every point at infinitely many scales, and hence is nowhere rectifiable.

The set  $E$  comes with a covering by the nested collections of intervals  $\mathcal{I}_k$ . The Hausdorff measure of  $E$  can be bounded from below in the usual way of defining a measure on  $E$  by distributing mass from one of these intervals to

all its children equally. Since  $nh(\delta/n) \rightarrow \infty$  for any fixed  $\delta > 0$  as  $n \rightarrow \infty$ , by choosing  $n_k$  large enough we can insure that

$$\sum h(|I_j|) \geq 2h(|I|),$$

where the sum is over the  $\mathcal{I}_{k+1}$  of  $I \in \mathcal{I}_k$ . A standard argument then shows that  $E$  has infinite  $h$ -measure.

## 7. INCREASING DIMENSION ON MANY LINES

We now prove the sharpness of the dimension bounds in Theorem 3.6.

Let  $E_\alpha$  be the Cantor set given by a standard iterative construction that starts with  $I_0 = [0, 1]$  and replaces each  $n$ th generation interval  $I$  with two  $(n+1)$ st generation intervals of length  $\alpha|I|$  and distance  $\frac{1}{3}(1-2\alpha)|I|$  from each other and from the endpoints of  $I$ . It is easy to check that  $\dim(E) = t = -\log 2 / \log \alpha$  and that  $t$  takes all values between 0 and 1 as  $\alpha$  goes from 0 to  $\frac{1}{2}$ . We also require that  $\alpha^{-k}$  is an integer for some positive integer  $k$  (which may depend on  $\alpha$ ). We claim it suffices to prove Theorem 3.6 in this case. If the theorem holds for all such  $\alpha$ 's, then for any  $d \in (0, 1)$  and  $\epsilon > 0$  we can choose such an  $\alpha$  and a  $f$  so that

$$\begin{aligned} t = \dim(E_\alpha) &> d, \\ \inf_{y \in E} \dim(f([0, 1] \times \{y\})) &\geq \frac{2}{t+1} - \frac{1}{2}\epsilon > \frac{2}{d+1} - \epsilon, \\ \inf_{0 \leq x \leq 1} \dim(f(x + iF)) &\geq \frac{2\dim(F)}{t+1} - \frac{1}{2}\epsilon > \frac{2\dim(F)}{d+1} - \epsilon, \end{aligned}$$

for any  $F \subset E_\alpha$ . Taking any  $F \subset E_\alpha$  with dimension  $d$  then proves the theorem.

So we may assume  $\alpha^{-k} = N$ . By taking multiples of  $k$ , we may assume  $N$  is as large as we wish. The basic building block is a quasiconformal map  $f$  from the square  $Q = [0, 1]^2$  to itself. Let  $\{I_j\}$ ,  $j = 1, \dots, 2^k$  be the  $k$ th generation covering intervals of  $E_\alpha$  and let  $R_j = [0, 1] \times I_j$  be rectangles with their short edges on the vertical sides of  $Q$ . Note that each of these is isometric to a  $1 \times \frac{1}{N}$  rectangle  $R$ . The map  $f$  will be conformal on each of these rectangles and will be quasiconformal on the rest of  $Q$ . Our construction also gives that  $f$  is the identity on the top and bottom edges of  $Q$  and it is symmetric with respect to the vertical bisector of  $Q$ , so  $f(1, y) = f(0, y) + 1$  on the vertical sides of  $Q$ . This means that  $f$  can be extended to a quasiconformal map of the whole plane by simply mapping each square  $Q + n + im$  to itself by  $(x, y) \rightarrow f(x - n, y - m) + n + im$ .

The map  $f$  is constructed by specifying a generalized quadrilateral  $T \subset Q$  (T for “tube”) that has two opposite sides on the vertical sides of  $Q$  and conformal modulus  $N$  (the same as  $R$ ). This means there is a conformal map  $\phi$  of  $R$  to  $T$  that maps vertices to vertices. This map is used to define a conformal map of each  $R_j$  to a translate  $T_j$  of  $T$  that connects the left and right sides of  $Q$ . The tubes  $T_j$  will have disjoint closures that do not hit the top and bottom edges of  $Q$  and so the complement of these tubes in  $Q$

are  $2^k + 1$  regions. We define a quasiconformal map from each component of  $Q \setminus \cup_j R_j$  to the corresponding component of  $Q \setminus \cup_j T_j$  that extends the mapping on each  $R_j$ , is the identity on the top and bottom edges of  $Q$  and is symmetric on the vertical edges of  $Q$ . The quasiconstant  $K$  of this map depends on the geometry of and spacing between the  $T_j$ , but is finite for the examples we will build.

The tube  $T$  will be constructed so that

$$|\phi'| \geq C_1(2\alpha)^{-k/2},$$

on all of  $R$  and with some constant  $C_1 > 0$  that is independent of  $k$ . First we use this estimate to finish the proof of the theorem, and then we construct a tube for which this estimate is true.

Let  $f_1 = f$ . The rectangle  $R$  has an obvious decomposition into  $N$  squares of side length  $1/n$ . Define a map  $f_2 : Q \rightarrow Q$  as the identity outside  $\cup_j R_j$  and inside each  $R_j$  use a scaled version of  $f$  to map each subsquare of  $R_j$  to itself. In general,  $f_n : Q \rightarrow Q$  is defined as the identity off the  $2^{nk}$  rectangles corresponding to the  $n$ th generation intervals covering  $E_\alpha$  and is defined using a scaled copy of  $f$  on the  $N^n$  squares making up each such rectangle. Then  $g_n = f_1 \circ \dots \circ f_n$  is quasiconformal with constant  $K$  (at most one map in the composition is non-conformal when applied to any point), so the limiting map is also  $K$ -quasiconformal.

For a fixed point  $y \in E$ , the segment  $S = [0, 1] \times \{y\}$  is covered by  $N^n$   $n$ th generation squares of side length  $N^{-n}$ , whose images all have diameters

$$\geq N^{-n}(C_1(2\alpha)^{-k/2})^n = C_1^n N^{-n/2} 2^{-kn/2} = \exp(n[\log C_1 - \frac{1}{2} \log N - \frac{k}{2} \log 2]).$$

Giving each image equal mass gives a measure  $\mu$  on  $f(S)$  that satisfies

$$\mu(f(Q)) = N^{-n} = \exp(-n \log N) \leq \text{diam}(f(Q))^s$$

where

$$\begin{aligned} s &= \frac{\log N}{-\log C_1 + \frac{1}{2} \log N + \frac{k}{2} \log 2} \\ &= \frac{-k \log \alpha}{-\log C_1 - \frac{k}{2} \log \alpha + \frac{k}{2} \log 2} \\ &= \frac{-\log \alpha}{-\frac{1}{k} \log C_1 - \frac{1}{2} \log \alpha + \frac{1}{2} \log 2} \\ &\rightarrow \frac{-\log \alpha}{-\frac{1}{2} \log \alpha + \frac{1}{2} \log 2} \\ &= \frac{2}{1 - \frac{\log 2}{\log \alpha}} \\ &= \frac{2}{1 + t}. \end{aligned}$$

Thus for  $k$  large enough,  $\mu$  satisfies

$$\mu(f(Q)) = N^{-n} \leq \text{diam}(f(Q))^{\frac{2}{1+t}-\epsilon}$$

for every  $n$ th generation covering square  $Q$  of  $S$ . This implies  $\dim(S) \geq \frac{2}{1+t} - \epsilon$  by the mass distribution principle, which is the first desired inequality.

For a fixed  $x \in [0, 1]$  then set  $x + iE$  is covered by  $2^k$   $n$ th generation squares  $Q$  of side length  $N^{-n}$ . If we give each equal mass and map this measure forward by  $f$ , then the image measure  $\nu$  satisfies

$$\nu(f(Q)) = 2^{-n} = \exp(-n \log 2) \leq \text{diam}(f(Q))^S$$

where  $S$  is given by (only the numerator is different from above)

$$\begin{aligned} S &= \frac{\log 2^k}{-\log C_1 + \frac{1}{2} \log N + \frac{k}{2} \log 2} \\ &= \frac{k \log 2}{-\log C_1 - \frac{k}{2} \log \alpha + \frac{k}{2} \log 2} \\ &= \frac{\log 2}{-\frac{1}{k} \log C_1 - \frac{1}{2} \log \alpha + \frac{1}{2} \log 2} \\ &\rightarrow \frac{\log 2}{-\frac{1}{2} \log \alpha + \frac{1}{2} \log 2} \\ &= \frac{-2 \log 2 / \log \alpha}{1 - \log 2 / \log \alpha} \\ &= \frac{2t}{1+t}. \end{aligned}$$

and hence  $\dim(f(x + iE)) > \frac{2t}{1+t} - \epsilon$  if  $k$  is large enough (depending on  $s$  and  $\epsilon$ ). This is the second claim of the theorem.

To deduce the final claim of the theorem, we recall that the tubes  $T_j$  in our construction have disjoint closures and hence are a positive distance apart. Thus the tubes in the  $n$ th generation of the construction are at least distance  $C_2^n N^{-n}$  apart. This means that if points  $z, w \in x + iE_\alpha$  are in different  $n$ th generation tubes, but are in the same  $(n-1)$ st generation tube, then

$$\begin{aligned} N^{-n} &\leq |z - w| \leq N^{-n+1}, \\ |f(z) - f(w)| &\geq C_2^n (N^{-n})^s, \end{aligned}$$

where  $s$  is as above. This implies the inverse of Hölder of order  $\frac{1}{s}$  which implies

$$\dim(f(E)) \geq s \cdot \dim(E),$$

for any subset  $E \subset x + iE_\alpha$ . This is the final conclusion of the theorem.

We have now done, except to build a tube  $T$  that has the proper estimate on the conformal map to a rectangle.

As before, let  $N = \alpha^{-k}$ . The  $1 \times \frac{1}{N}$  rectangle  $R$  divides into  $N$  disjoint squares and there are  $2^k$  such rectangles. Since  $\alpha < \frac{1}{2}$ , we have  $N \gg 2^k$  when  $k$  is large. Let

$$m = \left\lfloor \sqrt{\frac{N-1}{2^{k-1}}} \right\rfloor,$$

so that

$$\frac{1}{2}N \leq M = m^2 2^{k-1} + 1 \leq N,$$

if  $N$  is large enough.

Consider the  $m \times (2^k m + 1)$  rectangle  $W$  shown in Figure 6. The shaded squares form a tube  $T$  that connects the two vertical sides of the rectangle and uses  $M$  subsquares. If we think of  $T$  as a generalized quadrilateral with two sides (the “short sides”) on the vertical sides of  $W$  and two other sides (the “long sides”) that connect the vertical sides of  $W$ . We claim the extremal length  $\lambda$  of the path family connecting the shorts sides of  $T$  inside  $T$  satisfies

$$\frac{1}{2}M \leq M - 2^k m \leq \lambda \leq M.$$

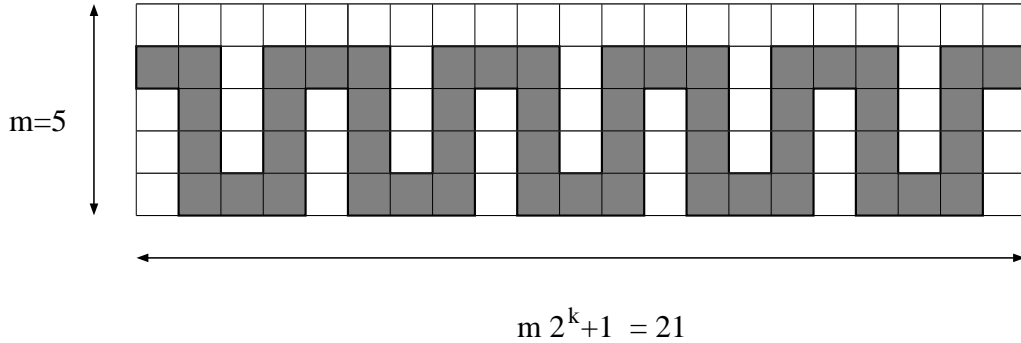


FIGURE 6. The first version of the tube is made up of  $M = m^2 2^{k-1} + 1$  squares forming a connected subset of a  $m \times (2^k m + 1)$  grid. Here we have taken  $m = 5$  and  $k = 2$ .

To prove the right hand side, take the constant metric 1 on  $T$ . Then every path connecting the long sides of  $T$  has length  $\geq 1$  for this metric and this metric has area  $M$ . Thus the extremal length of this family is  $\geq 1/M$  and since this family is conjugate to the one we want, we deduce  $\lambda \leq M$ . To prove the left hand inequality, take the metric  $\rho$  that is one on all the “non-corner” squares and zero elsewhere. If  $P$  is the number in non-corner squares then the  $\rho$ -length of every curve connecting the short sides of  $T$  is at least  $P$  and the area of  $\rho$  is  $P^2$  and hence  $\lambda \geq P$ . Since there are only  $2^k m$  corner squares, the left hand side follows.

The region  $T$  is almost the tube we want, but it is convenient to “round the corners” as shown in Figure 7. The long sides of the rounded tube are still distance 1 apart, so the extremal distance between the short sides is

still  $\lambda \leq M$ . In the proof of the lower bound, we set the metric to be zero on the corners and on squares adjacent to corners; this gives

$$\lambda \geq M - 32^k m \geq M \frac{m-6}{m} \geq \frac{1}{2}M$$

if  $m \geq 12$ . Rounding the corners implies that the conformal map of a rectangle  $R$  (of the same modulus as  $T$ ) to  $T$  can have derivative that is everywhere comparable to ratio of the widths of  $R$  and  $T$ .

The rounding is not actually necessary. If we leave the corners then the derivative of the conformal map will tend to zero or  $\infty$  at the corners, but will have uniform bounds on any subregion that is bounded away from the corners. The next generation of the construction will take place inside such a sub-region, and so the proof of the theorem would work even without rounding the corners (however, rounding is easy and gives a cleaner estimate).

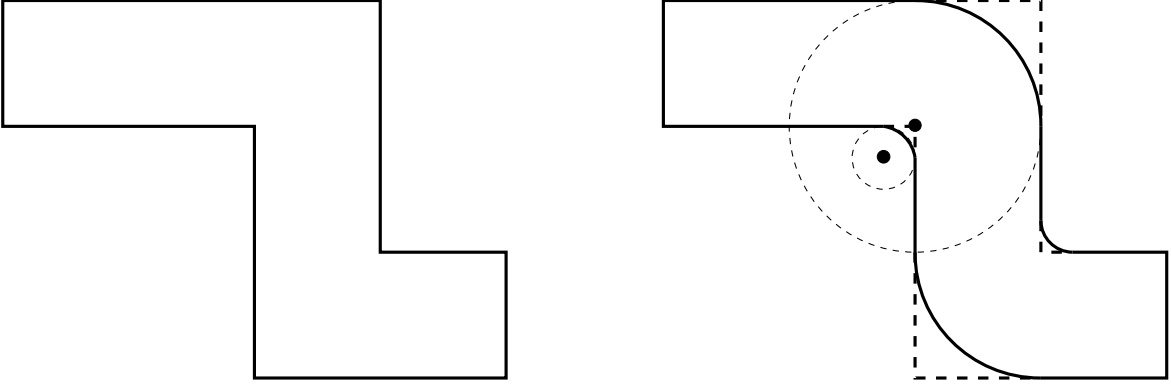


FIGURE 7. The second version of the tube has its non-vertex corners rounded off to insure the conformal map to the tube from the rectangle of the same modulus has a “nice derivative” (the minimum and maximum expansion have bounded ratio).

One more modification to  $T$  is needed. We have shown the extremal distance between the short sides of  $T$  is between  $\frac{1}{2}M$  and  $M \leq N$ , but we want it to be exactly  $N$ . We can make the extremal distance of  $T$  larger by making  $T$  “thinner” as illustrated on Figure 8. Since the extremal distance changes continuously with this perturbation of the domain, and can be made as large as we wish, there is a thinner version of  $T$  where the extremal distance between the short sides is exactly  $N$ . Since the extremal length starts  $\geq M/2 \geq N/4$ , the width of this thinner tube is comparable to the width of the original.

Let  $T$  denote the final rounded, thinned tube. Take the unit square  $Q = [0, 1]^2$  and subdivide it into  $(2^k m + 1)^2$  disjoint subsquares of side length  $(2^k m + 1)^{-1}$ , grouped into  $2^k$  rectangles of dimension  $m \times (2^k m + 1)$



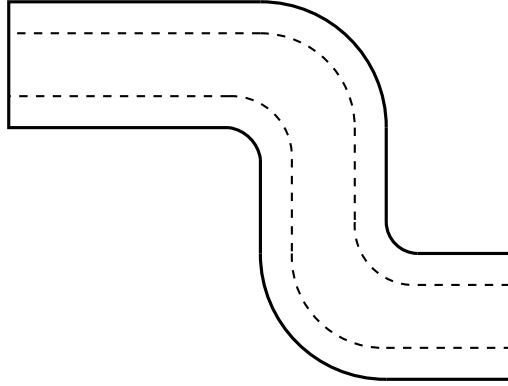


FIGURE 8. This version of the tube is narrower; this increases the extremal distance between the two short edges of the tube. By “thinning” the tube we can make its modulus exactly what we want. Since we start within a factor of 4 of the desired modulus, only a bounded amount of thinning is needed.

and the single  $1 \times (2^k m + 1)$  strip at the bottom. See Figure 9. Inside each rectangle, place a copy of the rounded, thinned tube  $T \subset T_0$ . These are the  $T_j$ ’s described earlier. The width of each  $T_j$  is comparable to

$$(2^k m + 1)^{-1} \simeq 2^{-k}/m \simeq 2^{-k} N^{-1/2} 2^{k/2} = 2^{-k/2} \alpha^{k/2},$$

and the width of each  $R_j$  is  $\alpha^k$ , so the conformal map from  $R_j$  to  $T_j$  (mapping vertices to vertices) has derivative everywhere comparable to

$$\frac{\text{width}(T_j)}{\text{width}(R_j)} = \frac{2^{-k/2} \alpha^{k/2}}{\alpha^k} = (2\alpha)^{-k/2},$$

as desired. This completes the proof of the theorem.

## 8. REMARKS AND QUESTIONS

**8.1. Estimates on the size of exceptional translates in  $\mathbb{R}^N$ .** If one considers “non-vertical” translates of a set  $E \subset \mathbb{R}^N$  then Theorem 3.3 still gives the following.

**Theorem 8.1.** *Let  $E \subset \mathbb{R}^N$  be an Ahlfors  $d$ -regular set and  $d' > d$ . Then for every quasisymmetric mapping  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$  the following holds*

$$(8.1) \quad \text{mod}_{\frac{N}{d}}(\{E + y : \dim_H f(E + y) > d'\}, \mathcal{H}_{\frac{d}{d}N}) = 0.$$

*In particular, for almost every set  $E + y$  we have*

$$(8.2) \quad \dim f(E + y) \leq \dim E.$$

One should be warned here, that the last statement of this theorem does not mean that the collection of  $y$ ’s such that  $\dim f(E + y) > \dim E$  has zero

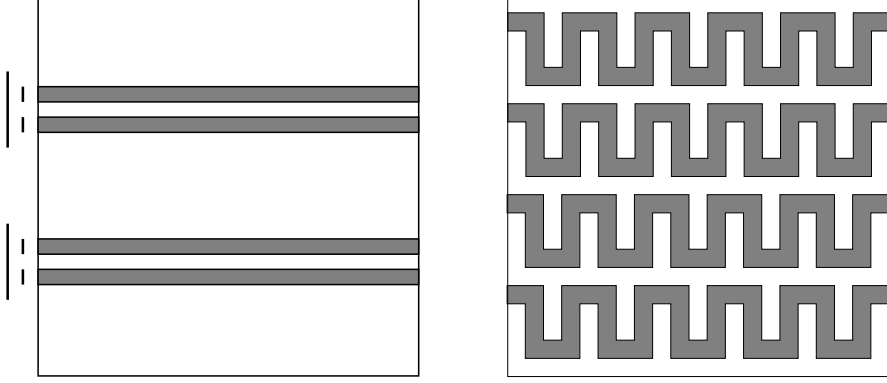


FIGURE 9. On the left is  $Q$  and the shaded rectangles  $R_j$ . On the right are the tubes  $T_j$  (we have omitted the rounding and thinning to make the picture simpler). Each  $R_j$  is conformally mapped to the corresponding  $T_j$  (they have the same modulus, so we can send vertices to vertices) and we quasiconformally extend these maps to map  $Q \rightarrow Q$  that is the identity on the top and bottom edges and is symmetric on the left and right edges (the tubes are symmetric with respect to the vertical bisector of  $Q$ , so this is possible).

measure, but rather that the corresponding family has zero Fuglede modulus with respect to  $\mathcal{H}_N$ .

In view of Theorem 8.1 the following seems to be a natural question to ask for “non-vertical” translates of a set  $E$ .

**Question 8.2.** *Let  $E \subset \mathbb{R}^n$ ,  $\dim_H E = d$ ,  $d < d' < N$  and  $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$  a quasiconformal mapping. Is*

$$\dim_H \{y \in \mathbb{R}^N : \dim_H f(E + y) > d'\} \leq \frac{d}{d'} \cdot N?$$

*In particular, is the following true*

$$\mathcal{H}_N(\{y \in \mathbb{R}^N : \dim_H f(E + y) > \dim_H E\}) = 0?$$

The answer is “yes” if  $\dim_H E$  is 0,  $N$  or if  $E$  is an affine subspace of  $\mathbb{R}^N$ . It seems this is not known for any  $E$  with non-integer Hausdorff dimension.

The examples of quasiconformal mappings which distort many lines in this paper are all two dimensional. Is the same true in higher dimensions?

**Question 8.3.** *Suppose  $N \geq 3$ .*

- *Is there a set  $E \subset \mathbb{R}^1$  of Hausdorff dimension  $N - 1$  and a QC map  $f$  of  $\mathbb{R}^N$  such that  $f(\mathbb{R} \times E)$  does not contain rectifiable subarcs?*
- *For  $\delta > 1$  and  $\varepsilon > 0$  is there a set  $E \subset \mathbb{R}^1$  of Hausdorff dimension  $(N/\delta - 1) - \varepsilon$  and a QC map  $f$  of  $\mathbb{R}^N$  such that  $f(\mathbb{R} \times E)$  does not contain subarcs of Hausdorff dimension  $< \delta$ ?*

**8.2. “K”-dependent results.** If  $E$  is of Hausdorff dimension 1 and if  $f$  is a  $K$  quasiconformal mapping of the plane then it was shown by Prause [21] that  $\dim_H f(E) \geq 1 - k^2$ , assuming  $f(\mathbb{R}) = \mathbb{R}$ . Here, as usual  $k = \frac{K-1}{K+1}$ . Recently, Astala, Prause and Smirnov showed in [4] that the same estimate holds for all quasiconformal maps of the plane, i.e.  $\dim_H f(E) \geq 1 - k^2$ , whenever  $f$  is  $K$ -quasiconformal and  $E \subset \mathbb{R}$  is of Hausdorff dimension 1. Combining this with Theorem 1.1 we obtain the following.

**Theorem 8.4.** *Suppose  $E \subset \mathbb{R}$  satisfies condition (3.11) with  $d = 1$ . Then for every Borel set  $Y \subset i\mathbb{R}$  and every  $K$ -quasiconformal mapping  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the following holds*

$$(8.3) \quad \frac{\dim_H f(E \times Y)}{\dim_H(E \times Y)} \geq 1 - \left( \frac{K-1}{K+1} \right)^2.$$

This is not sharp. If  $\dim_H(E \times [0, 1]) = 2$  then  $\dim_H f(E \times [0, 1]) = 2$  but the bound above gives a lower bound smaller than 2.

The following question may also be found in [5].

**Question 8.5.** *Let  $\mathcal{QC}(K)$  be the collection of  $K$ -QC maps of  $\mathbb{R}^2$ . Find*

$$(8.4) \quad \sup_{f \in \mathcal{QC}(K)} \dim\{y \in \mathbb{R} : f(\mathbb{R} + iy) \geq d'\}.$$

In [22] Smirnov proved that Hausdorff dimension of a  $K$ -quasicircle is at most  $1 + k^2$ . So in view of Theorem 8.4 the following seems to be a natural question.

**Question 8.6.** *Is it true that for a  $K$ -quasiconformal map  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  the following holds*

$$\frac{\dim_H f(\mathbb{R} \times Y)}{\dim_H \mathbb{R} \times Y} \leq 1 + \left( \frac{K-1}{K+1} \right)^2?$$

*Or more generally, is it true that for every 1-dimensional set  $E \subset \mathbb{R}$  and every Borel  $Y \subset \mathbb{R}$*

$$(8.5) \quad 1 - k^2 \leq \frac{\dim_H f(E \times Y)}{\dim_H E \times Y} \leq 1 + k^2,$$

*for every  $K$ -quasiconformal mapping  $f$  of the plane?*

**8.3. Exceptional sets for Sobolev quasiconformal maps.** If we assume that  $f : \mathbb{R}^N \rightarrow Y$  is not only quasisymmetric but also belongs to the so called Ambrosio-Reshetnyak-Sobolev space  $W^{1,q}(\mathbb{R}^N, Y)$  (see [3] and [5] for the definitions) then Theorem 3.3 together with a result of Kaufman from [16] gives better estimates on  $\dim_H \{y \in (\mathbb{R}^n)^\perp : \dim_H f(E) \geq d'\}$ . We will formulate only a corollary of a theorem of Kaufman, see Proposition 2.5 from [5].

**Lemma 8.7.** *Let  $X \subset \mathbb{R}^N$  be a set of  $\sigma$ -finite  $\mathcal{H}_t$  measure for some  $t \in (0, N)$  and  $f \in W^{1,q}(\mathbb{R}^N, Y)$  for some  $q > N$ . Then  $\dim_H f(X) \leq \frac{qt}{q-N+t}$ .*

**Theorem 8.8.** *Assume the conditions of Theorem 3.3 hold. If  $f$  is also in  $W^{1,q}(\mathbb{R}^N, Y)$  with  $q > N$  then*

$$(8.6) \quad \text{mod}_{\frac{p}{d}}(\{E + y : \dim_H f(E + y) > d'\}, \mu) = 0,$$

whenever  $\mu$  is upper  $p$ -regular with  $t > \frac{d}{d'}N$ .

Moreover, for “vertical” translates we have

$$\dim_H \{y \in (\mathbb{R}^n)^\perp : \dim_H f(E) \geq d'\} \leq \min\{(N - d) - \left(1 - \frac{d}{d'}\right)q, N - n\}.$$

*Proof.* The first part is an immediate consequence of Theorem 3.3 and Kaufman’s lemma with  $t = N$ .

For the second part suppose that  $Y = \{y \in (\mathbb{R}^n)^\perp : \dim_H f(E) \geq d'\}$  and let  $\beta = \dim_H Y$ . Then denoting  $X = E \times K$  we have  $\dim_X = \beta + d$ . By applying Theorem 1.1 to  $E \times Y$  we obtain

$$\frac{\beta + d}{d} \leq \frac{\dim_H f(X)}{d'} \leq \frac{1}{d'} \frac{q(\beta + d)}{q - N + \beta + d}.$$

Simplifying this inequality yields  $\beta \leq (N - d) + (1 - d/d')q$ .  $\square$

In [5] Sobolev mappings were considered and estimates for the size of exceptional families of parallel affine spaces were obtained. It would be interesting to know if the results of this paper also hold for such mappings.

**Question 8.9.** *Are Theorems 3.3, 3.1 and 8.8 true for  $f \in W^{1,q}(\mathbb{R}^N, Y)$  ( $f$  is not necessarily QC), where  $Y$  is a separable metric space?*

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